Journal of Pure and Applied Algebra 70 (1991) 199–210
North-Holland

Continuity, boundedness, connectedness and the Lindelöf property for topological groups*

F. Javier Trigos-Arrieta

Mathematics Department, Wesleyan University, Middletown, CT 06457, USA

Received 11 November 1989
Revised 21 March 1990

Abstract

Trigos-Arrieta, F.J., Continuity, boundedness, connectedness and the Lindelöf property for topological groups, Journal of Pure and Applied Algebra 70 (1991) 199–210.

If X is a space and $B \subseteq X$, we say that B is *functionally bounded* in X if every continuous real-valued function on X is bounded on B. For a locally compact Abelian group G with character group $\hat{G}$, we denote by G^+ the underlying group G equipped with the weakest topology which makes every $\chi \in \hat{G}$ continuous. For G as above we prove the following: (a) If $F \subseteq G$ then F is Lindelöf (functionally bounded) in G if and only if F is Lindelöf (functionally bounded) in G^+, (b) G is connected (zero-dimensional) if and only if G^+ is connected (zero-dimensional), and (c) If G and H are locally compact Abelian groups and $\phi : G \to H$ is an homomorphism then $\phi : G \to H$ is continuous if and only if $\phi : G^+ \to H^+$ is continuous. We generalize (c) to a result involving k-spaces.

0. Introduction and notation

0.1. Let G be a locally compact abelian group ($G \in$ LCA) and $\hat{G}$ its character group. Denote by G^+ the underlying group G with the weakest topology that makes every $\chi \in \hat{G}$ continuous. In [5, 3.4.3], [8], [12], [15, 3] and [19] it is shown that a subset of G is compact as a subspace of G if and only if it is compact as a subspace of G^+. We say then that G *respects compactness*. Also in [19] it is proved that the statement above remains true if we replace 'compact' by 'pseudocompact'. If X is a space and F one of its subsets, we say that F is *functionally bounded* in X if the

* This paper was presented by the author in the Conference of Locales and Topological Groups held in Curaçao in August 1989.

restriction to F of every continuous real-valued function of X is bounded. One result achieved here is the following: the above statement remains true if we replace 'compact' by 'Lindelöf' or 'functionally bounded'. Our proof depends heavily on a result due to E.K. van Douwen. We also give a necessary and sufficient condition for the continuity of an homomorphism between LCA groups in terms of their $+$-images, we see a connection between LCA groups and k-spaces, and we study the relation between G and G^+ in terms of connectedness and zero-dimensionality: G is connected (zero-dimensional) if and only if G^+ is connected (zero-dimensional).

0.2. Notation. All groups considered are abelian and all spaces are completely regular and Hausdorff. $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{T}$ denote the integers, real numbers and the unit circle respectively. If A is a set, $|A|$ will denote its cardinality. ω represents both the first infinite ordinal and the set of finite ordinals. ω_1 is the set of countable ordinals. If A is a nonempty subset of a group G, then $\langle A\rangle$ denotes the subgroup of G generated by A.

Let G be a group. We denote the identity element of G by 1_G. If G is a topological group, $N_G(1_G)$ denotes the base of neighborhoods of 1_G in G. If G is discrete we write $G^{\#}$ instead of G^+.

If X is a space, $C(X)$ and $C^*(X)$ denote the (rings of) continuous real-valued functions and continuous bounded real-valued functions from X into $\mathbb{R}$, respectively. Suppose now that $\{F_\alpha\colon \alpha\in A\}$ is a collection of disjoint sets. We use the notation $\dot{\bigcup}_{\alpha\in A} F_\alpha$ to denote the union of the F_α's and stress the fact that the F_α's are pairwise disjoint. Finally, if $F\subseteq X$, we denote by $\mathrm{Cl}_X F$ the closure of F in X.

1. Continuous homomorphisms

1.1. Let G be a (T_0) LCA group and $\hat{G}$ its group of continuous characters. We define an homomorphism e from G into the product of circles $\mathbb{T}^{\hat{G}}$ as follows:

$$e(g)_h = h(g).$$

Because $\hat{G}$ separates points [9, 23.26], e is a one-one, continuous homomorphism. We denote by G^+ the image of G under e as a subspace of $\mathbb{T}^{\hat{G}}$. Then G^+ is a totally bounded, completely regular, topological group. We identify the elements of the groups G and G^+.

Our goal in this section is to prove the following statement:

1.2. Theorem. *Let G and H be LCA groups and let $\phi: G\to H$ be a homomorphism. Then $\phi: G\to H$ is continuous if and only if $\phi: G^+\to H^+$ is continuous.*

We start out with three lemmas:

1.3. Lemma. *If G is a LCA group, H is a totally bounded abelian group and*

$\phi: G \to H$ is an algebraic homomorphism, then $\phi: G \to H$ is continuous if and only if $\phi: G^+ \to H$ is continuous.

Proof. ($\Rightarrow$) Let $\bar{H}$ be the Weil completion of H [2, 1.13], and view H as a subspace of $\mathbb{T}^{\bar{H}^\wedge}$. If $\psi \in \bar{H}^\wedge$ then $\psi \circ \phi \in \hat{G}$. Hence $\psi \circ \phi$ is continuous on G^+ for all $\psi \in \bar{H}^\wedge$ Hence $\phi: G^+ \to H$ is continuous.

($\Leftarrow$) $e: G \to G^+$ is continuous. Therefore $\phi = \phi \circ e$ is continuous. □

1.4. Lemma. *If G and H are* LCA *groups and $\phi: G \to H$ is a continuous homomorphism, then $\phi: G^+ \to H^+$ is continuous.*

Proof. $e: H \to H^+$ is continuous. Hence $\phi: G \to H^+$ is continuous. Now H^+ is totally bounded, so Lemma 1.3 applies. □

1.5. Lemma. *If G and H are* LCA *groups and $\phi: G^+ \to H^+$ is a continuous homomorphism, then the adjoint map $\hat{\phi}: \hat{H}^+ \to \hat{G}^+$ defined by*

$$\hat{\phi}(\chi)(g) = \chi(\phi(g))$$

is a well defined and continuous homomorphism.

Proof. The Weil completion of G^+ is the Bohr compactification bG of G. Since $\phi: G^+ \to H^+ \subseteq (H^+)^- = bH$ is uniformly continuous, there exists a continuous extension $\Phi: bG \to bH$. $\hat{\phi}$ is well defined because $\hat{\phi} = \hat{\Phi}$: $(bH)^\wedge (= (\hat{H})_{\mathrm{d}}) \to (bG)^\wedge$ $(= (\hat{G})_{\mathrm{d}})$ (the subscript $_{\mathrm{d}}$ means discrete) [9, 23.17, 24.38 and 26.12]. Now we want to show that $\hat{\phi}: \hat{H}^+ \to \hat{G}^+$ is continuous. Let $g \in \hat{G}^\wedge = G$. Then $g \circ \hat{\phi}(\chi) = \hat{\phi}(\chi)(g) = \chi(\phi(g)) = \phi(g)(\chi)$ for all $\chi \in \hat{H}$. It follows that $g \circ \hat{\phi} = \phi(g) \in \hat{H}^\wedge = H$. Hence $g \circ \hat{\phi}$ is continuous on H^+ for all $g \in \hat{G}^\wedge$. A function $\hat{\phi}$ into a product $\mathbb{T}^G$ is continuous if and only if each component $\pi_g \circ \hat{\phi}$ $(= g)$ is continuous. Therefore our result follows. □

1.6. Proof of Theorem 1.2. ($\Rightarrow$) Lemma 1.4.

($\Leftarrow$) Suppose $U \in N_H(1_H)$. We recall the following fact. If $K \subseteq G$ is compact and $\varepsilon > 0$, let $\mathbf{P}(K, \varepsilon) = \{\psi \in \hat{G}: |\psi(g) - 1| < \varepsilon$, for all $g \in K\}$. The family

$$\mathfrak{J} = \{\mathbf{P}(K, \varepsilon): K \subseteq G \text{ compact and } \varepsilon > 0\}$$

forms a neighborhood base for $1_{\hat{G}}$ in $\hat{G}$ [9, 23.15].

Now consider $\hat{\phi}$ defined as in Lemma 1.5. Let $K \subseteq \hat{H}$ and $\varepsilon > 0$ such that $\mathbf{P}(K, \varepsilon) \subseteq U$. Now K is compact in $\hat{H}^+$, hence $\hat{\phi}[K]$ is compact in $\hat{G}^+$ and therefore also in $\hat{G}$ (see 0.1, [8, 1.2] or [19, 1.4]). Let $V = \mathbf{P}(\hat{\phi}[K], \varepsilon)$. Then $V \in N_G(1_G)$ and it is not difficult to see that $\phi[V] \subseteq U$. So ϕ is continuous. □

1.7. Remark. Theorem 1.2 strengthens [8, 2.1] which is ($\Leftarrow$) above. In fact, Glicksberg's proof is quite different from ours. He starts out with a net converging to 1_G in a compact neighborhood of G and then, using the fact that LCA groups respect compactness, he proves that its image under ϕ converges to 1_H.

1.8. Remark. In 1.1 we can start more generally with maximal almost periodic abelian groups (MAP). This class contains LCA. For details see [10, VII]. Then Lemmas 1.3 and 1.4, with LCA replaced by MAP, remain valid with the same proofs. Now let us look closely at the proofs of Theorem 1.2 and Lemma 1.5. Let $\mathscr{P}$ be the class of groups satisfying Pontryagin duality [21, p. 592] and let $\mathfrak{R}$ be the class of groups that respect compactness, and set $\mathfrak{J}=\mathscr{P}\cap\mathfrak{R}$. We have that LCA $\subseteq\mathfrak{J}$ and this contention is proper: $\mathscr{P}$ is multiplicative [13], the Bohr compactification of a product is the product of the Bohr compactifications of the factors [4, 11]. Hence, as was pointed out to me by Comfort, any infinite product of noncompact LCA groups is in $\mathfrak{J}$ and is not in LCA. See [17] for details. Thus, by using the modified versions of Lemmas 1.3 and 1.4 above, we achieve a nontrivial generalization of Theorem 1.2 (and Lemma 1.5) by replacing LCA by $\mathfrak{J}$. We also note the following fact: in [21] the assertion is made that $\mathscr{P}\subseteq\mathfrak{R}$. In [17] is shown that *this is not correct*; for example, the (additive) topological group of an infinite-dimensional real Hilbert space satisfies Pontryagin duality, but does not respect compactness. So we might ask if Theorem 1.2 and Lemma 1.5 remain valid when LCA is replaced by $\mathscr{P}$ or $\mathfrak{R}$.

1.9. Definition. Let G be a totally bounded abelian group. We say that G is a *Glicksberg group*, and we write $G\in[G]$, if there exists a LCA group Γ such that $G=\Gamma^+$.

1.10. K.H. Hofmann remarked in conversation with the author that the category of k-spaces and continuous functions may be a reasonable and more natural setting for the arguments given in connection with Theorem 1.2. A space X is a *k-space* provided that for each $A\subseteq X$, the set A is open in X if the intersection of A with any compact subspace Z of the space X is open in Z. If X is any space, there is a unique k-space topology for X which is stronger than the original topology of X. The set X with this topology is denoted kX. The spaces X and kX have the same compact subspaces with the same topology on those subspaces and the identity function from kX onto X is continuous. We can see the topology on kX as the weakest one containing the topology of X. We refer to [6, pp. 201–204] for definitions and properties as well as the construction leading from an arbitrary Hausdorff space to a k-space with the same underlying set.

1.11. In [16] the concept of a *k-group* is introduced. Note that this concept is different from the one in [14]: There are k-groups in the sense of [14] which are not topological groups.

By [16, 1.1], for every topological group (G,t) there is a largest group topology $k_g(t)$ on G coinciding on compact sets with t. In the following we write $k_g(G)$ for the topological group $(G, k_g(t))$. Let t_k be the topology of kG. Then $k_g(t) \subseteq t_k$. Clearly both topologies coincide for locally compact groups.

1.12. Lemma. *Let G be a* LCA *group. Then* $k(G^+) = G = k_g(G^+)$.

Proof. G and G^+ have the same compact sets and G is a k-space. □

Our principal interest now is to find conditions to say when a totally bounded abelian group is a Glicksberg group. The following fact gives some light to the respect:

1.13. Lemma. $G \in [G] \Rightarrow k_g(G)$ *is a* LCA *group.* □

This follows immediately from Lemma 1.12. We do not know if the converse is true, i.e., if a totally bounded abelian group G such that kG is a LCA group has to be a Glicksberg group.

1.14. Example. In [14, 2.1] it is proved that if $\mathfrak{m}$ is an uncountable cardinal and $G = \mathbb{R}^{\mathfrak{m}}$, then kG is not a topological group. We now show an example of a totally bounded abelian group G such that kG is not a topological group either. With $\mathfrak{m}$ as above, $G = (\mathbb{R}^+)^{\mathfrak{m}}$ is totally bounded and it is not hard to see that $(\mathbb{R}^+)^{\mathfrak{m}}$ and $\mathbb{R}^{\mathfrak{m}}$ have the same compact subsets. Hence $k((\mathbb{R}^+)^{\mathfrak{m}}) = k(\mathbb{R}^{\mathfrak{m}})$ is not a topological group.

1.15. The following theorem answers a question suggested to the author by F.E.J. Linton. It improves somehow Theorem 1.2 by using Lemma 1.12.

Theorem. *Let G be a* LCA *group and H a totally bounded abelian group. Let $\phi : G \to H$ be a homomorphism from the group G into the group H. Then the following statements are equivalent*:

(a) $\phi : G^+ \to H$ *is continuous.*
(b) $\phi : G \to H$ *is continuous.*
(c) $\phi : G \to kH$ *is continuous.*
(d) $\phi : G \to k_g(H)$ *is continuous.*

Note that kH is just the k-space generated by H.

Proof. (a) $\Rightarrow$ (b). Obvious.

(b) $\Rightarrow$ (c). By Lemma 1.3 $\phi : G^+ \to H$ is continuous. [6, p. 204] implies that $\phi : k(G^+) \to kH$ is continuous. Now $k(G^+) = G$ by Lemma 1.12.

(c) $\Rightarrow$ (d). Obvious.

(d) $\Rightarrow$ (a). $\mathrm{id} : k_g(H) \to H$ is continuous. Hence $\phi : G \to H$ is continuous. Now use Lemma 1.3. □

Of course, if H is a Glicksberg group, Theorem 1.15 reduces to Theorem 1.2.

1.16. Using k-spaces we can also extend another result due to Glicksberg: if X is a locally compact space, G is a LCA group and $f: X \to G^+$ is a continuous *function*, then $f: X \to G$ is also continuous [8, 2.2]. The statement remains true if we just require X to be a k-space. This follows from Lemma 1.12 and the construction of a k-space given in [6].

1.17. Before closing this section we would like to raise a question about the topological structure of a LCA group G. Let G and H be LCA groups. By using [8, 2.2], the continuity of a function $f: G^+ \to H^+$ implies the continuity of $f: G \to H$. This also follows from Lemma 1.12. The converse, in general, is not true. Consider for example the discrete groups $\mathbb{Z}$ and $\mathbb{Q}$ (the set of rational numbers). If $f: \mathbb{Z} \to \mathbb{Q}$ is a bijective function which takes odd numbers into $[0,1] \cap \mathbb{Q}$ and even numbers into $\mathbb{Q} \setminus [0,1]$, trivially f is a homeomorphism between $\mathbb{Z}$ and $\mathbb{Q}$ but neither $f: \mathbb{Z}^+ \to \mathbb{Q}^+$ nor $f^{\leftarrow}: \mathbb{Q}^+ \to \mathbb{Z}^+$ is continuous, as the reader can easily check. However it is unknown whether the spaces $\mathbb{Z}^+$ $(=\mathbb{Z}^\#)$ and $\mathbb{Q}^+$ $(=\mathbb{Q}^\#)$ are homeomorphic:

Question (van Douwen). Let G and H be abelian groups such that $|G| = |H|$. Are the spaces $G^\#$ and $H^\#$ homeomorphic?

Note that, if we consider the groups G and H to be discrete, then G and H are LCA groups and the statement $|G| = |H|$ is equivalent to the statement that the spaces G and H are homeomorphic. We can extend the question above as follows:

Question. Let G and H be LCA groups such that the spaces G and H are homeomorphic. Are the spaces G^+ and H^+ homeomorphic?

Note that if G^+ and H^+ are homeomorphic then the spaces G and H are homeomorphic, by [8, 2.2] or Lemma 1.12.

1.18. The remarks and comments in 1.8 and 1.11, as well as the equality $G = k_g(G^+)$ in Lemma 1.12 and the inclusion of (d) in Theorem 1.15 were kindly pointed out to the author by the referee. D. Dikranjan has pointed out that another proof of Theorem 1.2 is available in [5, 3.4.5].

2. Connectedness

2.1. Our interest here is to determine how conditions like connectedness or zero-dimensionality are preserved under passage between G and G^+. If $G \in \mathrm{LCA}$ and H is a locally compact subgroup of G then H is closed in G [9, 5.11]. From this, it follows that H is closed in G^+ [19, 3.3] because H is the intersection of kernels of continuous characters [9, 24.12]. Hence the groups G/H, G^+/H and $(G/H)^+$ are Tychonoff.

2.2. Lemma. *Let G and H be* LCA *groups such that $H \subseteq G$. Then* $\mathrm{id}: G^+/H \to (G/H)^+$ *is a topological isomorphism.*

Proof. For continuity it is enough to show that every $\chi \in (G/H)^\wedge$ is continuous as a function on G^+/H. Fix $\chi \in (G/H)^\wedge$ and let $\varepsilon > 0$ be given. Then $N_\varepsilon(1) = \{x \in \mathbb{T} : |x-1| < \varepsilon\}$ is a basic neighborhood of 1 in $\mathbb{T}$. If $\phi : G \to G/H$ and $\phi^+ : G^+ \to G^+/H$ denote the natural maps, then $\chi \circ \phi \in G^\wedge$, so it is continuous on G^+. Hence $\phi^+[(\chi \circ \phi)^\leftarrow[N_\varepsilon(1)]]$ is open in G^+/H. But $\phi^+[(\chi \circ \phi)^\leftarrow[N_\varepsilon(1)]] = \chi^\leftarrow[N_\varepsilon(1)]$. Therefore χ is continuous on G^+/H, as required.

Now we prove that $\mathrm{id}: (G/H)^+ \to G^+/H$ is continuous. Since $\mathrm{id}: G \to G^+$ is continuous, the same is true for $\mathrm{id}: G/H \to G^+/H$. Now apply Lemma 1.3. $\square$

2.3. Theorem. *If G is a* LCA *group which is zero-dimensional, then G^+ is zero-dimensional.*

Proof. Let $U \in N_{G^+}(1_G)$, and choose $W \in N_{G^+}(1_G)$ such that $W^2 \subseteq U$. Plainly W is open in G. Choose a compact open subgroup $H \subseteq W$ [9, 7.7], and let $\phi : G \to G/H$ denote the natural map. G/H is discrete, so by Lemma 2.2 the group $G^+/H = (G/H)^\#$ is zero-dimensional ([3, 2.2] or [20, 4.8]). Now choose a closed and open neighborhood V of H in $(G/H)^\#$ such that $V \subseteq \phi[W]$. Then $\phi^\leftarrow[V] \subseteq U$ because $W^2 \subseteq U$, and $\phi^\leftarrow[V]$ is closed and open in G^+. $\square$

2.4. Theorem. *Let G be a* LCA *group, C the component of 1_G in G, and D the component of 1_G in G^+. Then $C = D$.*

Proof. Because $e: G \to G^+$ is continuous, we have that $C = e[C] \subseteq D$. Now G/C is a LCA zero-dimensional group ([9, 7.3] and [22, 29.7]). Hence $G^+/C = (G/C)^+$ is also zero-dimensional, by Lemma 2.2 and Theorem 2.3. But this implies that $C = D$ because no subset of G^+ containing C properly can be connected. $\square$

2.5. Remark. The referee has contributed the following alternative proof of Theorem 2.4:

(1) Let G be a LCA group. If G^+ is connected, then G is also connected: Since G^+ is connected, $h[G]$ is connected for all $h \in (G^+)^\wedge$. By [1, p. 18(c)], h is surjective if $h \neq 1$. Since $(G^+)^\wedge = G^\wedge$, [1, 8.2] implies that G is connected. (2) Clearly $C \subseteq D$. D is a closed subgroup of G^+, hence also of G. Thus D is a LCA subgroup of G. By [19, 3.4], D (as a subspace of G^+) is the same as D^+. Thus D^+ is connected. Now (1) implies that D (as a subgroup of G) is connected too. So $D \subseteq C$. This completes the proof. $\square$

2.6. Combining Theorems 2.3 and 2.4 we get the following:

Corollary. *Let G be a* LCA *group.*

(a) *G is connected if and only if G^+ is connected.*
(b) *G is zero-dimensional if and only if G^+ is zero-dimensional.* □

3. The Lindelöf property

3.1. This section is devoted to a proof of the following:

Theorem. *Let G be a* LCA *group. Let F be a subset of G. F is Lindelöf as a subspace of G if and only if F is Lindelöf as a subspace of G^+.*

Theorem 3.1 answers a question posed to the author in conversation by L.C. Robertson.

3.2. We use the following remarkable result of van Douwen [20, 1.3].

Theorem. *Let G be a discrete abelian group of infinite cardinality. Let F be an infinite subset of G. There exists $D \subseteq F$ with $|D| = |F|$ such that D is relatively discrete and C-embedded in $G^{\#}$.* □

3.3. Lemma (Comfort). *If E is an uncountable space with a dense, relatively discrete, $\mathbb{C}$-embedded subset D such that $|D| = |E|$, then E is not Lindelöf.*

Proof. If E were Lindelöf then it is realcompact [7, 8.2] and the fact that D is $\mathbb{C}$-embedded in E implies that $E = \upsilon D$ (the *Hewitt-Nachbin realcompactification* of D) [7, 8.7(II)]. Write $D = \dot{\bigcup}_{\xi<\omega_1} D_\xi$ with each $D_\xi \neq \emptyset$. Then $\upsilon D = \dot{\bigcup}_{\xi<\omega_1} \upsilon D_\xi$ [7, 12G] and each of the sets $\upsilon D_\xi = (\mathrm{Cl}_{\beta D} D_\xi) \cap \upsilon D$ is closed and open in υD [7, 8.7 and 6.9(a) and (c)]. Clearly the open cover $\{\upsilon D_\xi : \xi < \omega_1\}$ does not have a countable subcover, so υD is not Lindelöf. □

3.4. The 'discrete' version of Theorem 3.1 is the following:

Lemma. *If G is an abelian group and $F \subseteq G^{\#}$ is Lindelöf then F is countable.*

Proof. Suppose that $|F| > \omega$. Using Theorem 3.2 there exists $D \subseteq F$ such that $|D| = |F|$, D is relatively discrete and C-embedded in $G^{\#}$. Let $E = \mathrm{Cl}_F D$. If F is a Lindelöf space then so is E, contradicting Lemma 3.3. □

3.5. The σ-compact version of Theorem 3.1 is the following:

Lemma. *Let H be a* LCA, *σ-compact group. Suppose that $F \subseteq H^+$ is Lindelöf. Then $F \subseteq H$ is Lindelöf.*

Proof. Recall that if $K \subseteq H$ is compact in H or H^+, then $K \subseteq H$ and $K \subseteq H^+$ are homeomorphic spaces. Suppose $H = \bigcup_{n<\omega} K_n$ with each K_n compact. Plainly $F = \bigcup_{n<\omega} (F \cap K_n)$ and each $F \cap K_n$ is Lindelöf as a subspace of H^+. Because of the comment above, $F \cap K_n$ is Lindelöf as a subspace of H and therefore F, the countable union of Lindelöf spaces, itself has to be Lindelöf as a subspace of H. □

3.6. Proof of Theorem 3.1. ($\Rightarrow$) Obvious.

($\Leftarrow$) Use [9, 5.14] to find a LCA, σ-compact closed and open subgroup H of G. Then G/H is discrete and $(G/H)^+ = (G/H)^\#$. Using Lemma 3.4, F hits only countably many translates of H. If $xH \cap F \neq \emptyset$ for $x \in G$, we have that $xH \cap F$ is Lindelöf as a subspace of xH^+. This is because H is closed in G^+ and H^+ and H—as a subspace of G^+—are the same spaces [19, 3.3 and 3.4]. By Lemma 3.5, $xH \cap F$ is Lindelöf as a subspace of xH, hence of G. Being the countable union of Lindelöf spaces, F itself has to be Lindelöf. □

3.7. Remarks (Comfort). (a) We know [19] that pseudocompact subspaces of LCA groups are homeomorphic to their $+$-image under the inclusion function. This is not longer true for Lindelöf spaces. Consider $F = \mathbb{R} = G$. Here we have the following strange phenomenon: If $n < \omega$, denote by $[-n, n]^+$ the subspace $[-n, n]$ of $\mathbb{R}^+$ (which actually is just $[-n, n]$). Then $\mathrm{id}_n : [-n, n] \to [-n, n]^+$ is a homeomorphism. However, the limit function $\mathrm{id} : \mathbb{R} \to \mathbb{R}^+$ is not a homeomorphism. Hence, the union of a countable increasing sequence of homeomorphisms is not necessarily a homeomorphism.

(b) There are LCA groups with Lindelöf subspaces which are not σ-compact. Let X be any Lindelöf non σ-compact space. (For example if D is an uncountable discrete space, fix a point x_0 in D and if $x_0 \in U \subseteq D$, declare that U is open if $|D \setminus U| \leq \omega$, for the other points in D keep their local bases unaltered. The space X obtained in this fashion is clearly Lindelöf and not σ-compact.) X is Tychonoff and normal. Embed X into $\mathbb{T}^{C^*(X)}$ and let $G = \mathrm{Cl}_{\mathbb{T}^{C^*(X)}}\langle X \rangle$. Then G is a compact abelian group and X is a Lindelöf, non σ-compact subspace of G.

4. Boundedness on groups

4.1. Definition (Tkachenko [18]). Let X be a space. If $B \subseteq X$ we say that B is *functionally bounded in* X if every $f \in C(X)$ is bounded on B.

4.2. This section is devoted to proving the following result:

Theorem. *Let G be a* LCA *group and $B \subseteq G$. Then*:

(a) *B is functionally bounded in G if and only if B is functionally bounded in G^+.*

(b) *If this occurs then* $B \subseteq G$ *and* $B \subseteq G^+$ *are homeomorphic and* $\mathrm{Cl}_G B = \mathrm{Cl}_{G^+} B$. *Furthermore, the latter space is compact.*

4.3. It is important to mention that van Douwen's result Theorem 3.2 plays an important rôle in the discrete version of Theorem 4.2. We will also use the following trivial fact: If X and Y are spaces, $f: X \to Y$ is a continuous function and $B \subseteq X$ is functionally bounded in X then $f[F]$ is functionally bounded in Y.

4.4. The following is the discrete version of Theorem 4.2:

Lemma. *If* G *is a discrete abelian group then* $B \subseteq G$ *is functionally bounded in* $G^{\#}$ *if and only if* $|B| < \omega$.

Proof. If $|B| \geq \omega$, Theorem 3.2 assures the existence of some $D \subseteq B$ such that $|D| = |B|$, D relatively discrete and C-embedded in $G^{\#}$. This proves that if B is functionally bounded in $G^{\#}$ it must be finite. The converse is obvious. □

4.5. The following is the σ-compact version of Theorem 4.2:

Lemma. *If* G *is a* LCA, *σ-compact group and* $B \subseteq G$, *then* B *is functionally bounded in* G *if and only if* B *is functionally bounded in* G^+.

Proof. ($\Rightarrow$) Trivial.

($\Leftarrow$) Suppose B is functionally bounded in G^+. Plainly $F = \mathrm{Cl}_{G^+} B$ is functionally bounded in G^+. Note that G^+ is normal because G and thus G^+ are σ-compact. Therefore F is pseudocompact in G^+. By [19, 4.1], F is compact in G and hence any $f \in C(G)$ is bounded on F, therefore on B. So B is functionally bounded in G. □

4.6. Proof of Theorem 4.2. Let B be functionally bounded in G^+. Choose $n < \omega$ and a LCA group G_2 with a compact open subgroup H such that $G = \mathbb{R}^n \times G_2$ [9, 24.30]. Let $G_1 = \mathbb{R}^n$. If $\pi_i : G^+ \to G_i^+$ ($i = 1, 2$) are the projection maps, we have that $\pi_i[B]$ is bounded in G_i^+. Because G_1 is σ-compact, $\pi_1[B]$ is functionally bounded in $\mathbb{R}^n$ (Lemma 4.5). Hence $\mathrm{Cl}_{\mathbb{R}^n} \pi_1[B]$ is compact. Let $B_1 = \pi_2[B]$. By Lemma 4.4, B_1 hits only finitely many translates of the compact group H. Hence B_1 is functionally bounded in G_2 and $\mathrm{Cl}_{G_2} B_1$ is compact. Hence $B \subseteq \mathrm{Cl}_{\mathbb{R}^n} \pi_1[B] \times \mathrm{Cl}_{G_2} \pi_2[B]$, a compact subspace of G. Thus B is functionally bounded in G.

We just saw that any functionally bounded subspace B of G^+ has to be contained in a compact subspace K of G. K is compact in G^+ with the same topology. Therefore $\mathrm{Cl}_G B = \mathrm{Cl}_{G^+} B$ is compact and the last statement follows. □

4.7. By using [19] we get the following:

Corollary. *If G is a* LCA *group and $B \subseteq G$, then the following statements are equivalent*:

(a) *B is functionally bounded in G.*

(b) *B is functionally bounded in G^+.*

(c) $\mathrm{Cl}_G B$ *is pseudocompact.*

(d) $\mathrm{Cl}_{G^+} B$ *is pseudocompact.*

(e) $\mathrm{Cl}_G B$ *is compact.*

(f) $\mathrm{Cl}_{G^+} B$ *is compact.*

Further, when any one of the conditions holds the two spaces $\mathrm{Cl}_G B$, $\mathrm{Cl}_{G^+} B$ *are equal.* □

4.8. Remark. We just saw that the map $e: G \to G^+$ keeps unaltered the topology on bounded subspaces (a class which contains compact subspaces and even all pseudocompact subspaces). This is not true for Lindelöf subspaces. Also note that the space K constructed in [19, Section 2] is countable and discrete in both $\mathbb{R}$ and $\mathbb{R}^+$ and is functionally bounded in neither. If G is discrete, Theorem 3.2 shows that each infinite subspace of $G^\#$ always contains a discrete space of the same cardinality which trivially is homeomorphic to its preimage under e. So we may ask if it is possible to achieve a characterization of those subspaces X of LCA groups such that X in G and X in G^+ are homeomorphic.

Acknowledgment

The author is grateful to Professor Karl H. Hofmann, Professor Fred E.J. Linton and Professor Lewis C. Robertson for several helpful conversations on topics related to the results of this paper. It is a pleasure to express my deep gratitude to my teacher Professor W.W. Comfort for stimulating discussions about the subject and for a very careful reading of early versions of this paper. I am indebted to the referee for the nice report he made on a first version of this article. He improved it by generalizing many results, by bringing several relevant papers to my attention, and by giving alternate proofs and different points of view. I also use this opportunity to thank the Department of Mathematics of Wesleyan University and the organizers of the Curaçao Conference on Locales and Topological Groups for their hospitality and support.

References

[1] D.L. Armacost, The structure of locally compact abelian groups, Pure and Applied Mathematics Series 68 (Marcel Dekker, New York, 1981).

[2] W.W. Comfort, Topological groups, in: Kenneth Kunen and Jerry Vaughan, eds., Handbook of General Topology (North-Holland, Amsterdam, 1984) 1143–1263.

[3] W.W. Comfort and F.J. Trigos, Remarks on a Theorem of Glicksberg, 1990, Submitted for publication; See also: Abstracts Amer. Math. Soc. 9 (1988) 420–421 (= Abstract #88T-22-195).

[4] J. de Vries and M. Hušek, Preservation of products by functors close to reflectors, Topology Appl. 27 (1987) 171–189.

[5] D.N. Dikranjan, I.R. Prodanov and L.N. Stoyanov, Topological Groups, Pure and Applied Mathematics Series 130 (Marcel Dekker, New York, 1989).

[6] R. Engelking, General Topology, Polska Akademia Nauk, Monograf. Mat. 60 (PWN, Warsaw, 1977).

[7] L. Gillman and M. Jerison, Rings of Continuous Functions, Graduate Text in Mathematics 43 (Springer, New York, 1976).

[8] I. Glicksberg, Uniform boundedness for groups. Canad. Math. 14 (1962) 269–276.

[9] E. Hewitt and K.A. Ross, Abstract Harmonic Analysis, Volume I, Grundlehren der Mathematischen Wissenschaften 115 (Springer, Berlin, 1963).

[10] H. Heyer, Dualität lokalkompakter Gruppen, Lecture Notes in Mathematics 1150 (Springer, Berlin, 1970).

[11] P. Holm, On the Bohr compactification, Math. Ann. 156 (1964) 34–46.

[12] R. Hughes, Compactness in locally compact groups, Bull. Amer. Math. Soc. 79 (1973) 122–123.

[13] S. Kaplan, Extensions of Pontryagin duality I: Infinite products, Duke Math. J. 15 (1948) 649–658.

[14] W.F. LaMartin, On the foundations of k-group theory, Dissertationes Math. CXLVI, 1977.

[15] W. Moran, On almost periodic compactifications of locally compact groups, J. London Math. Soc. (2) 3 (1971) 507–512.

[16] N. Noble, k-groups and duality, Trans. Amer. Math. Soc. 151 (1970) 551–561.

[17] D. Remus and F.J. Trigos-Arrieta, Remarks on a paper of Venkataraman (title tentative), Manuscript in preparation.

[18] M.G. Tkachenko, Boundedness and pseudocompactness in topological groups, Mat. Zametki 41 (1987) 400–405 (in Russian); English translation: Math. Notes 41 (1987) 229–231.

[19] F.J. Trigos-Arrieta, Pseudocompactness on groups, Proceedings of the Northeast Conference on Topology and its Applications, 1989, to appear.

[20] E. van Douwen, The maximal totally bounded group topology on G and the biggest minimal G-space, for Abelian groups G, Topology Appl. 34 (1990) 69–91.

[21] R. Venkataraman, Compactness in abelian topological groups, Pacific J. Math. 57 (1975) 591–595.

[22] S. Willard, General Topology (Addison Wesley, Reading, MA, 1970).